Sp(2)-Symmetric Langrangian BRST Quantization

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Abstract

One Lagrangian BRST quantization principle is that of imposing correct Schwinger-Dyson equations through the BRST Ward identities. In this paper we show how to derive the analogous $Sp(2)$-symmetric quantization condition in flat coordinates from an underlying $Sp(2)$-symmetric Schwinger-Dyson BRST symmetry. We also show under what conditions this can be recast in the language of triplectic quantization.

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1 Introduction

One general principle with which one can derive the BRST quantization of Lagrangian field theories is that of imposing correct Schwinger-Dyson equations at the level of the BRST symmetry itself [1]. In conventional Lagrangian BRST quantization this can be phrased in terms of a general quantum Master Equation of the form [2]

$$\frac{1}{2}(S,S) = -\frac{\delta^r S}{\delta \phi^A} c^A + i\hbar \Delta S \ , \tag{1}$$

with a set of fields to be more accurately specified below, and the antibracket $(\cdot, \cdot)$ and Δ-operator of the Batalin-Vilkovisky formalism [3]. Care must be taken in order to impose appropriate boundary conditions on S, since otherwise correct Schwinger-Dyson equations will not be guaranteed. One proper choice amounts to the Ansatz $S[\phi, \phi^*, c] = S^{BV}[\phi, \phi^*] - \phi_A^* c^A$, for which this most general Master Equation (1) reduces to that of Batalin and Vilkovisky [3],

$$\frac{1}{2}(S^{BV}, S^{BV}) = i\hbar \Delta S^{BV} \ . \tag{2}$$

In order to derive a corresponding $Sp(2)$-symmetric BRST quantization principle, one must impose an $Sp(2)$-symmetric Schwinger-Dyson BRST symmetry [1, 4] on the action. One way of achieving this has been discussed in ref. [5]. Except for a few inessential technical details the result could be cast in a form proposed earlier on purely algebraic grounds by Batalin, Lavrov and Tyutin [6].

Recently, Batalin and Marnelius [7] have proposed a very interesting alternative $Sp(2)$-invariant Lagrangian BRST scheme which they call "triplectic quantization".[1] Like the Batalin-Vilkovisky formalism itself [3] and the the $Sp(2)$-symmetric analogue of ref. [6], the method of triplectic quantization is introduced in a most remarkable way. A certain Master Equation for the action is postulated (in this case actually two equations, as will be reviewed below), and it is *a posteriori* observed that solutions to this Master Equation will give actions that leave the path integral invariant under a particular $Sp(2)$ symmetric BRST-like symmetry.

While the formalism of Batalin, Lavrov and Tyutin already consists of an extension of the conventional fields ϕ^A to include "antifields" ϕ_{Aa}^* (here and in the following lower case Roman indices denote $Sp(2)$ indices) and certain new fields $\bar{\phi}_A$, triplectic quantization amounts even in its most minimal form to an additional extension with a further set of "antifields" π_a^A.[2] By extending the space of field variables in this manner, triplectic quantization is completely "anticanonical": to the pair of fields ϕ^A and $\bar{\phi}_A$ correspond a pair of canonically conjugate field variables ϕ_{Aa}^* and π_a^A within an $Sp(2)$-extended antibracket [7]

$$\{F,G\}^a \equiv \frac{\delta^r F}{\delta \phi^A}\frac{\delta^l G}{\delta \phi_{Aa}^*} + \frac{\delta^r F}{\delta \bar{\phi}_A}\frac{\delta^l G}{\delta \pi_a^A} - \frac{\delta^r F}{\delta \phi_{Aa}^*}\frac{\delta^l G}{\delta \phi^A} - \frac{\delta^r F}{\delta \pi_a^A}\frac{\delta^l G}{\delta \bar{\phi}_A} \ , \tag{3}$$

and with a corresponding $Sp(2)$-extended Δ-operator

$$\hat{\Delta}^a \equiv (-1)^{\epsilon_A+1}\left[\frac{\delta^r}{\delta \phi_{Aa}^*}\frac{\delta^r}{\delta \phi^A} + \frac{\delta^r}{\delta \pi_a^A}\frac{\delta^r}{\delta \bar{\phi}_A}\right] \ . \tag{4}$$

It was emphasized already in ref. [5] that especially in the $Sp(2)$-symmetric case there is a huge degree of arbitrariness in the choice of formalism. This arbitrariness shows up both in the choice of

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[1]Its covariant generalization is discussed in ref. [8]; see also ref. [9].

[2]Our distinction between "fields" and "antifields" differs from that of refs. [7, 8]. The reason for our choice of nomenclature will become obvious below.

what one considers the fundamental Master Equation, and how the gauge fixing is achieved. While it was possible in [5] to derive the formalism of Batalin, Lavrov and Tyutin [6] from the underlying principle of imposing Schwinger-Dyson BRST symmetry, it was stressed that many other different – but completely equivalent – formulations exist. Now, with the existence of a seemingly quite different formalism, that of triplectic quantization, this issue has taken on more urgency. Is the method of Batalin, Lavrov and Tyutin not as general as triplectic quantization? How do the two methods relate to each other? It is furthermore of interest to clarify the possible advantage or disadvantage in making a further extension of the space of fields in order to achieve a complete pairing in canonically conjugate fields. At a more technical level there are also several questions which have been raised by the recent work of ref. [7]. For example, the $Sp(2)$ BRST symmetry of triplectic quantization appears to depend crucially on the gauge fixing (denoted by X in ref. [7]) even in the most simple cases of closed irreducible gauge algebras. How can this be understood? Compared with the formulation of Batalin, Lavrov and Tyutin, we also note some interesting differences. Intuitively one could understand the need for two antifields ϕ_{Aa}^* in the formalism of ref. [6] as the need for having sources of both BRST symmetries δ_a, while the required additional field $\bar{\phi}_A$ could be understood from the need for having sources for the independent commutator $\frac{1}{2}\epsilon^{ab}\delta_a\delta_b$ as well. With an $Sp(2)$ BRST algebra satisfying $\{\delta_a, \delta_b\} = 0$ one might naively not expect the need for yet more antifields π_a^A, and it is of interest to have their origin clarified.

Another issue is the following. Since we argue that a complete Lagrangian quantization prescription can be derived from one underlying fundamental principle, that of Schwinger-Dyson BRST symmetry, it is interesting to see to what extent triplectic quantization can be derived from this principle. The purpose of this paper is to discuss such a derivation. Along the way we hope to clarify a number of puzzling points, including the precise relationship between the formalism of Batalin, Lavrov and Tyutin and that of triplectic quantization.

As our starting point, in the next section, we shall write down the most general Master Equation any action must satisfy in order to be invariant under $Sp(2)$-covariant Schwinger-Dyson BRST symmetry.[3] With a specific choice of boundary conditions this Master Equation reduces exactly to that of Batalin, Lavrov and Tyutin [6]. We next proceed, in section 3, with a reformulation of this scheme that fits directly into the framework of triplectic quantization, while simultaneously illustrating how one could invent a number of equivalent alternative formulations. As will become clear here, triplectic quantization encompasses all solutions based on the $Sp(2)$-covariant Schwinger-Dyson BRST symmetry, but will – depending on the boundary conditions – also go beyond. We trace this possible generalization of the space of solutions to a departure from the requirement of an $Sp(2)$-symmetric BRST algebra of the form $\delta_a\delta_b + \delta_b\delta_a = 0$. Section 4 of this paper is devoted to a brief summary of our findings, and some suggestions for future work.

Our discussion is restricted to functional integrals with a regulator that respects all pertinent symmetries of the original fields, or, alternatively, to the finite-dimensional case.

2   A General Lagrangian Sp(2) Master Equation

The fundamental symmetry on which we will base the quantization is that of an $Sp(2)$-symmetric version of the Schwinger-Dyson BRST symmetry. This particular variant can most conveniently, be derived within a collective field formalism. In ref. [5] it was shown how to accomplish this with (for each field ϕ^A) one collective field φ^A. It turns out to be more advantageous to use a formulation in terms of *two* collective fields $\varphi_{1,2}^A$. This idea was already outlined in ref. [10], but the resulting

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[3]Our discussion is always restricted to flat (or "Darboux") coordinates. The covariant case can be derived analogously, see later.

$Sp(2)$-invariant Schwinger-Dyson BRST symmetry turned out to be a twisted version of the more simple set of Schwinger-Dyson BRST transformations which we will derive below. A formulation using two collective fields also appears to be required for consistent superfield constructions [11].

Intuitively, the fact that two collective fields $\varphi_{1,2}^A$ provide a more convenient formulation can be understood from the fact that in the $Sp(2)$-covariant case we will need two BRST sources. The third source, for the commutator of the two transformations, will – as we shall see shortly – appear automatically. Note that φ_1^A and φ_2^A here are treated as two independent collective fields, not forming an $Sp(2)$ doublet.

Since the functional measures are assumed flat, we first make the simple transformation $\phi^A \to \phi^A - \varphi_1^A - \varphi_2^A$. Grassmann parities are here obviously $\epsilon(\phi^A) \equiv \epsilon_A = \epsilon(\varphi_1^A) = \epsilon(\varphi_2^A)$, and ϕ^A includes all classical fields as well as ghosts, antighosts, ghosts-for-ghosts, etc., in the usual manner. As many of the following steps have already been described in detail in refs. [2] and [5], we shall be rather brief at this point. Both the action S and the functional measure are invariant under the BRST transformations[4]

$$
\begin{aligned}
\delta_a \phi^A &= \pi_a^A \\
\delta_a \varphi_1^A &= \frac{1}{2}\pi_a^A + \frac{1}{2}\phi_a^{*A} - \frac{1}{2}\mathcal{R}_a^A[\phi - \varphi_+] \\
\delta_a \varphi_2^A &= \frac{1}{2}\pi_a^A - \frac{1}{2}\phi_a^{*A} - \frac{1}{2}\mathcal{R}_a^A[\phi - \varphi_+] \\
\delta_a \pi_b^A &= \epsilon_{ab} \lambda^A \\
\delta_a \phi_b^{*A} &= \epsilon_{ab} B^A \\
\delta_a \lambda^A &= \delta_a B^A = 0 \ ,
\end{aligned}
\tag{5}
$$

where it is convenient to introduce the notation $\varphi_\pm^A \equiv \varphi_1^A \pm \varphi_2^A$. Ghost number assignments are as follows:

$$
\begin{aligned}
\mathrm{gh}(\pi_a^A) &= \mathrm{gh}(\phi_a^{*A}) = (-1)^{a+1} + \mathrm{gh}(\phi^A) \\
\mathrm{gh}(\varphi_\pm^A) &= \mathrm{gh}(\lambda^A) = \mathrm{gh}(B^A) = \mathrm{gh}(\phi^A) \ ,
\end{aligned}
\tag{6}
$$

with the overall normalization fixed by the requirement that the action has ghost number zero. We are here illustrating the BRST symmetry for the simple case of a closed irreducible gauge algebra, returning to the most general case shortly. The detailed definition and properties of the objects $\mathcal{R}_a^A$ are not important for the present discussion. The only requirement is

$$
\frac{\delta^r \mathcal{R}_a^A}{\delta \phi^B} \mathcal{R}_b^B + \frac{\delta^r \mathcal{R}_b^A}{\delta \phi^B} \mathcal{R}_a^B = 0 \ .
\tag{7}
$$

Note that because of the huge redundancy of a description in terms of two additional collective fields, the action $S_0[\phi - \varphi_+]$ and the internal BRST generators (including the gauge generators) are independent of φ_-. In the case of no internal gauge symmetries ($\mathcal{R}_a^A = 0$), it is straightforward to check that the BRST Ward Identities associated with (5) are the most general Schwinger-Dyson equations once both collective fields have been gauge fixed to zero, and the additional ghost fields have been integrated out.

It is thus the *sum* of the two collective fields that plays the most active rôle in this formulation, while the difference φ_-^A decouples in the BRST algebra. This is particularly obvious if we rewrite the BRST rules in terms of these fields:

$$
\delta_a \phi^A = \pi_a^A
$$

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[4] $Sp(2)$-indices are raised and lowered by the ϵ-tensor.

$$\begin{aligned}
\delta_a \varphi_+^A &= \pi_a^A - \mathcal{R}_a^A[\phi - \varphi_+] \\
\delta_a \pi_b^A &= \epsilon_{ab} \lambda^A \\
\delta_a \lambda^A &= 0 \\
\delta_a \varphi_-^A &= \phi_a^{*A} \\
\delta_a \phi_b^{*A} &= \epsilon_{ab} B^A \\
\delta_a B^A &= 0 .
\end{aligned} \tag{8}$$

As compared with the analogous treatment in ref. [5], the last three equations are new, but they clearly represent a completely decoupled BRST multiplet. In this formulation, it is the ghost (antighost) π_a^A which is the $Sp(2)$ analogue of the c^A-ghost in conventional Lagrangian quantization [2].

We now gauge-fix in a conventional $Sp(2)$-symmetric manner both collective fields $\varphi_\pm^A$ to zero. Because the gauge-fixing term must be bosonic and have overall ghost number zero, we do this with the help of an invertible constant matrix M_{AB}, and add

$$\begin{aligned}
\frac{1}{2} \epsilon^{ab} \delta_a \delta_b [\varphi_+^A M_{AB} \varphi_-^B] &= -\varphi_+^A M_{AB} B^B + (-1)^{\epsilon_A+1} \epsilon^{ab} (\pi_a^A - \mathcal{R}_a^A) M_{AB} \phi_b^{*B} \\
&\quad - \lambda^A M_{AB} \varphi_-^B - \frac{1}{2} \epsilon^{ab} \frac{\delta^r \mathcal{R}_b^A}{\delta \phi^C} \mathcal{R}_a^C M_{AB} \varphi_-^B
\end{aligned} \tag{9}$$

to the action. Here the matrix M has grassmann parity $\epsilon(M_{AB}) = \epsilon_A + \epsilon_B$ and ghost number $\mathrm{gh}(M_{AB}) = -\mathrm{gh}(\phi^A) - \mathrm{gh}(\phi^B)$. We can denote the inverse of M by a raising of indices: $M_{AB} M^{BC} = \delta_A^C$. We now integrate out B^A. This removes φ_+^A from the path integral as well, and we see that we have to replace

$$B^A \rightarrow -M^{AB} \frac{\delta^l S}{\delta \phi^B} \tag{10}$$

in the BRST rules. Here S is the full action (without the 'gauge boson' term), not just the classical part $S_0[\phi]$. At this point it is clear that the matrix M, as expected, can be removed. We do this by a redefinition

$$\begin{aligned}
\bar{\phi}_A &\equiv (-1)^{\epsilon_A+1} M_{AB} \varphi_-^B \\
\phi_{Aa}^* &\equiv M_{AB} \phi_a^{*B} .
\end{aligned} \tag{11}$$

This changes ghost number assignments, so that now $\mathrm{gh}(\bar{\phi}_A) = -\mathrm{gh}(\phi^A)$ and $\mathrm{gh}(\phi_{Aa}^*) = (-1)^a - \mathrm{gh}(\phi^A)$. In terms of these variables the action takes the form

$$S = S_0[\phi] + \epsilon^{ab} \phi_{Ab}^* (\pi_a^A - \mathcal{R}_a^A) + \bar{\phi}_A \lambda^A + \frac{1}{2} \epsilon^{ab} \bar{\phi}_A \frac{\delta^r \mathcal{R}_b^A}{\delta \phi^C} \mathcal{R}_a^C , \tag{12}$$

and the BRST transformation rules are now

$$\begin{aligned}
\delta_a \phi^A &= \pi_a^A \\
\delta_a \bar{\phi}_A &= (-1)^{\epsilon_A+1} \phi_{Aa}^* \\
\delta_a \pi_b^A &= \epsilon_{ab} \lambda^A \\
\delta_a \phi_{Ab}^* &= -\epsilon_{ab} \frac{\delta^l S}{\delta \phi^A} \\
\delta_a \lambda^A &= 0 .
\end{aligned} \tag{13}$$

The quantization procedure is now essentially complete. All that remains is the addition of an $Sp(2)$-exact gauge-fixing term $S_\chi \equiv \frac{1}{2} \epsilon^{ab} \delta_a \delta_b \chi$ which fixes the internal gauge symmetry. So the full

action is $S_{\text{ext}} = S + S_\chi$. Of course, in this case the answer is known beforehand, and we have in any case restricted ourselves to the simple case of an irreducible closed gauge algebra.

The important observation now is that the Schwinger-Dyson BRST symmetry of eq. (13) is completely independent of any possible local gauge symmetries. At this point we are therefore able to derive the most general conditions any $Sp(2)$ BRST-symmetric action must fulfill. Namely, the first condition is that the Schwinger-Dyson BRST transformations (13) must be a symmetry of the path integral. The second is that if we allow for an extension of the action $S[\phi]$ to include, say, terms involving ϕ_{Aa}^* and $\bar{\phi}_A$, then these fields must be automatically integrated to zero in the path integral, before any gauge fixings of the internal symmetries. Finally, in order that Ward Identities of the form $0 = \langle \epsilon^{ab}\delta_b[\phi_{Aa}^* F(\phi)]\rangle$ become identical to the correct Schwinger-Dyson equations, we must require $\langle \pi_b^B \phi_{Aa}^* \rangle = -(i\hbar)\delta_A^B \epsilon_{ba}$.

The first of these conditions, that the path integral must be invariant under (13) means that the BRST-variation of the action, $\delta_a S_{\text{ext}}$, must cancel the Jacobian from the functional measure. In detail:

$$\delta_a S = \frac{\delta^r S}{\delta \phi^A}\pi_a^A - \frac{\delta^r S}{\delta \phi_{Ab}^*}\epsilon_{ab}\frac{\delta^l S}{\delta \phi^A} + \frac{\delta^r S}{\delta \bar{\phi}_A}(-1)^{\epsilon_A+1}\phi_{Aa}^* + \frac{\delta^r S}{\delta \pi_b^A}\epsilon_{ab}\lambda^A = (i\hbar)\epsilon_{ab}(-1)^{\epsilon_A+1}\frac{\delta^r \delta^r S}{\delta \phi_{Ab}^* \delta \phi^A} \quad (14)$$

To write it more compactly, we can introduce the $Sp(2)$ extended antibracket [6]

$$(F,G)^a \equiv \frac{\delta^r F}{\delta \phi^A}\frac{\delta^l G}{\delta \phi_{Aa}^*} - \frac{\delta^r F}{\delta \phi_{Aa}^*}\frac{\delta^l G}{\delta \phi^A} \ , \quad (15)$$

and the related Δ-operator [6],

$$\Delta^a \equiv (-1)^{\epsilon_A+1}\frac{\delta^r}{\delta \phi_{Aa}^*}\frac{\delta^r}{\delta \phi^A} \ . \quad (16)$$

The full quantum Master Equation (14) is then

$$\frac{1}{2}(S,S)^a + U^a S = i\hbar \Delta^a S \ , \quad (17)$$

where[5]

$$U^a F \equiv \epsilon^{ab}\frac{\delta^r F}{\delta \phi^A}\pi_b^A + (-1)^{\epsilon_A+1}\epsilon^{ab}\frac{\delta^r F}{\delta \bar{\phi}_A}\phi_{Ab}^* + \frac{\delta^r F}{\delta \pi_a^A}\lambda^A \ . \quad (18)$$

The two other conditions the action S must satisfy can both be met by the simple choice

$$S[\phi, \phi^*, \bar{\phi}, \pi, \lambda] = S^{BLT}[\phi, \phi^*, \bar{\phi}] + \epsilon^{ab}\phi_{Ab}^*\pi_a^A + \bar{\phi}_A\lambda^A \ , \quad (19)$$

and the first boundary condition is then $S^{BLT}[\phi, \phi^*=0, \bar{\phi}=0] = S_0[\phi]$. Inserting this into the full Quantum Master Equation (14), we recover the Master Equation of Batalin, Lavrov and Tyutin [6]:

$$\frac{1}{2}(S^{BLT}, S^{BLT})^a + V^a S^{BLT} = i\hbar \Delta^a S^{BLT} \ . \quad (20)$$

Here,

$$V^a F \equiv (-1)^{\epsilon_A+1}\epsilon^{ab}\frac{\delta^r F}{\delta \bar{\phi}_A}\phi_{Ab}^* \ . \quad (21)$$

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[5]Our definitions of both Δ-operators and vector fields systematically employ right-derivatives. This causes some differences in our definitions compared with refs. [3, 6, 7]. However, when acting on objects of even Grassmann parities (such as actions) our definitions agree. The right-derivatives occur naturally because we follow the convention of the Batalin-Vilkovisky formalism, where BRST variations act as right-derivatives. This becomes of importance when one considers, *e.g.*, quantum versions of the BRST operator (see section 2.1). Our conventions are described in detail in the appendix of ref. [2].

At this point we see the rôle played by the additional "antifields" π_a^A in this formulation: they are the $Sp(2)$ analogues of the additional c-ghosts of ref. [2]. They serve as to set the antifields ϕ_{Aa}^* equal to zero prior to any gauge fixing of internal symmetries. Similarly, the fields λ^A are Lagrange multipliers ensuring that the additional fields $\bar{\phi}_A$ vanish before any gauge fixings. For this reason these additional fields $\pi_a^A, \bar{\phi}_A$ are spectator fields with respect to the antibracket (15). Furthermore, they have completely dropped out from the Master Equation for S^{BLT}, as they should from self-consistency.[6]

The gauge-fixing procedure is now straightforward. Having found a solution S^{BLT} of eq. (20), we can add, for any bosonic function ("gauge boson") $\chi[\phi, \pi, \bar{\phi}, \lambda]$ of zero ghost number, a term $\frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi$ to the action. This follows from the fact that the $Sp(2)$ BRST algebra $\delta_a\delta_b + \delta_b\delta_a = 0$ is fulfilled on an arbitrary function of the above fields. This is the biggest advantage of the present formulation which is based on two collective fields, compared with the formulation of ref. [5] which was based on only one collective field. A term of the form $S_\chi[\phi, \pi, \bar{\phi}, \phi^*, \lambda] \equiv \frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi[\phi, \pi, \bar{\phi}, \lambda]$ is the most general gauge-fixing function that can be added to the solution S^{BLT} of the Master Equation (20).

As a special case we can of course restrict ourselves to gauge bosons that are functions only of the fields ϕ^A. This is what is done in ref. [6], and gauge fixing then simply amounts to adding

$$\frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi(\phi) = -\frac{\delta^r\chi}{\delta\phi^A}\lambda^A + \frac{1}{2}\epsilon^{ab}\pi_b^A\frac{\delta^r\delta^r\chi}{\delta\phi^B\delta\phi^A}\pi_a^B \qquad (22)$$

to the action.

If one insists, one can try to interpret the special case above as a condition on how to "replace" the antifields ϕ_{Aa}^* and the fields $\bar{\phi}_A$ in the path integral after gauge fixing. As we see, the direct replacement works only for $\bar{\phi}_A$, where the rule is that it should be replaced by $\delta^r\chi/\delta\phi^A$. Due to the quadratic term involving π_a^A, the condition on ϕ_{Aa}^* is not directly of a substitution-type. One can of course linearize this quadratic π_a^A-term at the cost of introducing yet more redundant degrees of freedom through new "antifields" λ_{Aa}^*. Gauge-fixing then amounts to simple substitutions for the antifields ϕ_{Aa}^* as well (somewhat reminiscent of the conventional Batalin-Vilkovisky formalism), but there are then no simple substitution rules for the λ_{Aa}^* fields. It seems more convenient to simply add the gauge-fixing function to the action, and perform the required functional integrals directly.

Remarkably, the $Sp(2)$-covariant version of Schwinger-Dyson BRST symmetry (13) was observed, without derivation, in ref. [6] to hold for the solution of the Master Equation after one particular gauge fixing, when the fields π_a^A and λ^A were suitably introduced to parametrize the gauge-fixing function. The importance of this symmetry beyond gauge-fixing, and its relation to Schwinger-Dyson BRST symmetry was however not noted.

We have already mentioned that the Schwinger-Dyson BRST symmetry (13) is not unique in providing correct Schwinger-Dyson equations as $Sp(2)$-symmetric BRST Ward Identities. One choice with a fewer number of fields has already been considered in ref. [5]. Another choice, discussed in ref. [10], employs the same BRST multiplet of fields, but corresponds to a twisting of

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[6]In the covariant case, away from flat coordinates, this leads to non-trivial conditions on the transformation law for the antifields, as described in ref. [13] for the analogous case without $Sp(2)$ symmetry.

the transformations. Explicitly, it can be written in the form

$$
\begin{array}{rclcrcl}
\delta_1\phi^A &=& \pi_1^A & \qquad & \delta_2\phi^A &=& \pi_2^A \\
\delta_1\varphi_1^A &=& \pi_1^A - \phi_2^{*A} & & \delta_2\varphi_1^A &=& -\phi_1^{*A} \\
\delta_1\varphi_2^A &=& \phi_2^{*A} & & \delta_2\varphi_2^A &=& \pi_2^A + \phi_1^{*A} \\
\delta_1\pi_1^A &=& 0 & & \delta_2\pi_1^A &=& -\lambda^A \\
\delta_1\pi_2^A &=& \lambda^A & & \delta_2\pi_2^A &=& 0 \\
\delta_1\lambda^A &=& 0 & & \delta_2\lambda^A &=& 0 \\
\delta_1\phi_1^{*A} &=& B^A - \tfrac{1}{2}\lambda^A & & \delta_2\phi_1^{*A} &=& 0 \\
\delta_1\phi_2^{*A} &=& 0 & & \delta_2\phi_2^{*A} &=& -B^A - \tfrac{1}{2}\lambda^A \\
\delta_1 B^A &=& 0 & & \delta_2 B^A &=& 0
\end{array}
\tag{23}
$$

It is straightforward to check that this also satisfies the correct $Sp(2)$ algebra $\delta_a\delta_b + \delta_b\delta_a = 0$. It is reassuring that this different version of the Schwinger-Dyson BRST transformation rules leads to precisely the same quantum Master Equation (20). The required algebra is, however, slightly tedious, and since it leads to precisely the same equations we will not show any details here.

2.1 Sp(2)-covariant Quantum BRST

In conventional Batalin-Vilkovisky Lagrangian quantization the BRST operator is not just the action itself (acting within the antibracket), it also contains a "quantum correction" [14]. In the language of ref. [2] the BRST operator is completely classical, but if one integrates out the ghosts c^A of that formulation, one recovers the quantum correction in terms of the remaining fields. It is tempting to see what an analogous $Sp(2)$-covariant quantum BRST operator could look like. According to the lessons learned from the conventional Batalin-Vilkovisky formalism, this should correspond to a formulation in which the π_a^A-fields have been integrated out.

We shall restrict ourselves to the Ansatz (19), and first consider the fate of the BRST symmetry when integrating out the π_a^A-fields *before* any gauge-fixing term S_χ has been added. We shall use the identity

$$
\int [d\pi] F[\pi_c^C] \exp\left\{ \frac{i}{\hbar}\epsilon^{ab}\phi_{Ab}^*\pi_a^A \right\} = F\left[(i\hbar)\epsilon_{cd}\frac{\delta^l}{\delta\phi_{Cd}^*} \right] \int [d\pi] \exp\left\{ \frac{i}{\hbar}\epsilon^{ab}\phi_{Ab}^*\pi_a^A \right\} . \tag{24}
$$

Consider the BRST-variation of an arbitrary function $G = G(\phi, \phi^*, \bar{\phi}, \lambda)$:

$$
\delta_a G = \frac{\delta^r G}{\delta\phi^A}\pi_a^A + \frac{\delta^r G}{\delta\bar{\phi}_A}(-1)^{\epsilon_A+1}\phi_{Aa}^* + \frac{\delta^r G}{\delta\phi_{Ab}^*}\left(-\epsilon_{ab}\frac{\delta^l S}{\delta\phi^A} \right) . \tag{25}
$$

Inside the functional integral we allow ourselves to perform partial integrations. Consider then the first term of eq. (25). Inside the path integral the following manipulations are valid:

$$
\begin{aligned}
\int [d\pi] \frac{\delta^r G}{\delta\phi^A}\pi_a^A \exp\left\{ \frac{i}{\hbar}\epsilon^{cb}\phi_{Bb}^*\pi_c^B \right\} &= \frac{\delta^r G}{\delta\phi^A}(i\hbar)\epsilon_{ab}\left[\frac{\delta^l}{\delta\phi_{Ab}^*}\delta(\phi^*) \right] \\
\rightarrow \quad & -\left[(-1)^{(\epsilon_A+1)(\epsilon_A+\epsilon_G)}(i\hbar)\epsilon_{ab}\frac{\delta^l}{\delta\phi_{Ab}^*}\left[\frac{\delta^r G}{\delta\phi^A} \right] + \frac{\delta^r G}{\delta\phi^A}(i\hbar)\epsilon_{ab}\left(\frac{i}{\hbar} \right)\frac{\delta^l S^{BLT}}{\delta\phi_{Ab}^*} \right]\delta(\phi^*) \\
&= \left[(-1)^{\epsilon_A}(i\hbar)\epsilon_{ab}\left[\frac{\delta^r}{\delta\phi_{Ab}^*}\frac{\delta^r}{\delta\phi^A}G \right] + \frac{\delta^r G}{\delta\phi^A}\epsilon_{ab}\frac{\delta^l S^{BLT}}{\delta\phi_{Ab}^*} \right]\delta(\phi^*) ,
\end{aligned}
\tag{26}
$$

where the arrow has indicated when a partial integration has been performed.

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In total, including the additional terms of eq. (25), we get, for an arbitrary function $G[\phi, \phi^*, \bar\phi, \lambda]$:

$$\delta_a G \rightarrow -(i\hbar)\Delta_a G + \frac{\delta^r G}{\delta\phi^A}\epsilon_{ab}\frac{\delta^l S^{BLT}}{\delta\phi^*_{Ab}} + \frac{\delta^r G}{\delta\bar\phi_A}(-1)^{\epsilon_A+1}\phi^*_{Aa} - \frac{\delta^r G}{\delta\phi^*_{Ab}}\epsilon_{ab}\frac{\delta^l S^{BLT}}{\delta\phi^A} \ . \tag{27}$$

As one could have hoped, the action S^{BLT} is (almost) the full BRST operator within the $Sp(2)$-covariant antibracket once the π-fields have been integrated out. There is also an additonal term generated by the vector field V_a, and a "quantum correction" generated by Δ_a. In detail, one sees that the $Sp(2)$ "quantum BRST operator" σ_a is defined by

$$\sigma_a \equiv (\ \cdot\ , S^{BLT})_a + V_a - (i\hbar)\Delta_a \ . \tag{28}$$

When a gauge-fixing term S_χ is added, the total action becomes more than linear in the π-fields (when the "gauge boson" χ is chosen to be a function of ϕ^A alone, the action becomes quadratic in the π-fields), and the integral over π^A_a no longer results in a δ-function constraint on ϕ^*_{Aa}. Although the simple derivation above then does not go through, the quantum BRST operator σ_a nevertheless remains unchanged. The easiest way to see this may be the following. Split up the action S_{ext}:

$$S_{\text{ext}} = S^{BLT}[\phi, \phi^*, \bar\phi] + X^{BLT} \ , \tag{29}$$

where

$$X^{BLT} \equiv \epsilon^{ab}\phi^*_{Ab}\pi^A_a + \bar\phi_A\lambda^A + S_\chi[\phi, \pi, \lambda] \ , \tag{30}$$

and where for simplicity we have restricted ourselves to gauge-fixing functions S_χ arising from gauge bosons $\chi = \chi[\phi, \pi, \lambda]$. In terms of these variables our Schwinger-Dyson BRST symmetry can be written

$$\begin{aligned}
\delta^a\phi^A &= -\frac{\delta^l X^{BLT}}{\delta\phi^*_{Aa}} \\
\delta^a\bar\phi_A &= (-1)^{\epsilon_A+1}\epsilon^{ab}\phi^*_{Ab} \\
\delta^a\pi^A_b &= \delta^a_b\frac{\delta^l X^{BLT}}{\delta\bar\phi_A} \\
\delta^a\phi^*_{Ab} &= -\delta^a_b\frac{\delta^l S^{BLT}}{\delta\phi^A} \\
\delta^a\lambda^A &= 0 \ .
\end{aligned} \tag{31}$$

Consider now in all generality the way δ^a acts on an arbitrary function $G(\phi, \bar\phi, \phi^*, \pi, \lambda)$:

$$\begin{aligned}
\delta^a G &= -\frac{\delta^r G}{\delta\phi^A}\frac{\delta^l X^{BLT}}{\delta\phi^*_{Aa}} + \frac{\delta^r G}{\delta\bar\phi_A}(-1)^{\epsilon_A+1}\epsilon^{ab}\phi^*_{Ab} + \frac{\delta^r G}{\delta\pi^A_a}\frac{\delta^l X^{BLT}}{\delta\bar\phi_A} - \frac{\delta^r G}{\delta\phi^A}\frac{\delta^l S^{BLT}}{\delta\phi^A} \\
&= -\frac{\delta^r G}{\delta\phi^*_{Aa}}\frac{\delta^l S^{BLT}}{\delta\phi^A} - \frac{\delta^r G}{\delta\phi^A}\frac{\delta^l X^{BLT}}{\delta\phi^*_{Aa}} + V^a G + \frac{\delta^r G}{\delta\pi^A_a}\frac{\delta^l X^{BLT}}{\delta\bar\phi_A} \ .
\end{aligned} \tag{32}$$

Now focus on the 2nd term in eq. (32). By a partial integration we have:

$$\frac{\delta^r G}{\delta\phi^A}\frac{\delta^l X^{BLT}}{\delta\phi^*_{Aa}} \rightarrow -\frac{\delta^r G}{\delta\phi^A}\frac{\delta^l S^{BLT}}{\delta\phi^*_{Aa}} + (-1)^{\epsilon_A+1}(i\hbar)\frac{\delta^r}{\delta\phi^*_{Aa}}\frac{\delta^r}{\delta\phi^A}G \ . \tag{33}$$

Inserting this into eq. (32) we find, in all generality:

$$\delta^a G \rightarrow (G, S^{BLT})^a + V^a G - (i\hbar)\Delta^a G + \frac{\delta^r G}{\delta\pi^A_a}\frac{\delta^l X^{BLT}}{\delta\bar\phi_A}$$

$$= \quad \sigma^a G + \frac{\delta^r G}{\delta \pi_a^A} \frac{\delta^l X^{BLT}}{\delta \bar{\phi}_A} \ . \tag{34}$$

For the gauge-fixing function under consideration we can simply replace $\delta^l X^{BLT}/\delta \bar{\phi}_A$ by λ^A.

The above manipulations are not changed if the π-fields are integrated out. The left-over action is of course appropriately modified, and it is then only meaningful to consider Green functions $G = G(\phi, \bar{\phi}, \phi^*, \lambda)$ which do not depend on π_a^A. The $Sp(2)$ BRST operator δ^a is then precisely replaced by the $Sp(2)$ quantum BRST operator σ^a. If one integrates out λ^A as well, the result is again unchanged, but the analysis is then of course restricted to $G = G(\phi, \bar{\phi}, \phi^*)$.

It is straighforward to verify that this quantum BRST operator satisfies the $Sp(2)$-invariant algebra

$$\sigma^a \sigma^b + \sigma^b \sigma^a = 0 \tag{35}$$

when acting on arbitrary functions of all fields. As in the conventional Lagrangian BRST case, nilpotency is thus recovered when the shift-ghosts (here π_a^A) are integrated out. One must use the quantum Master Equation for S^{BLT} in order to show this. Because of the Δ^a-part, the quantum BRST operator σ^a fails to act like a derivation.

2.2 KEEPING THE COLLECTIVE FIELDS

So far our approach has been to treat the collective fields only as tools with which one can derive the required symmetries. In the present formulation based on two collective fields, we removed the sum of these two fields, while we kept their difference (which became the new field $\bar{\phi}_A$ through the linear redefinition (11)). This is the most compact formulation which fulfills all our requirements. In particular, the $Sp(2)$-symmetric analogue of nilpotency of the BRST operator(s), $\delta_a \delta_b + \delta_b \delta_a = 0$, remains valid when acting on all fields except the antifields ϕ_{Aa}^*.

Conceptually it may, however, be advantageous to keep the collective fields all the way to the end of the quantization procedure. One obvious advantage is that nilpotency of the BRST operator is retained on all fields and antifields. To give an example, consider conventional Batalin-Vilkovisky quantization. When the collective fields are integrated out, nilpotency holds only when the BRST operator acts on functions of the fields ϕ^A (and c^A). Gauge-fixing is then done by adding the BRST-variation of a gauge fermion,

$$\delta \Psi(\phi) = \frac{\delta^r \Psi(\phi)}{\delta \phi^A} c^A \ , \tag{36}$$

which, to make contact with the original Batalin-Vilkovisky prescription [3], is taken to depend on the fields ϕ^A only. This results in a δ-function constraint in the functional integral, which replaces the antifield ϕ_A^* by $\delta^r \Psi(\phi)/\delta \phi^A$. If instead one keeps the collective field, one can choose to add the BRST-variation of a more general gauge fixing fermion $\Psi = \Psi(\phi, \phi^*)$[7]. Instead of (36), one has in that case

$$\delta \Psi(\phi, \phi^*) = \frac{\delta^r \Psi}{\delta \phi^A} c^A + \frac{\delta^r \Psi}{\delta \phi_A^*} B_A \ . \tag{37}$$

The upshot is that upon integrating out the collective field it is not fixed to zero now, but to the derivative of the gauge fermion w.r.t. the antifield. In terms of gauge fixings of the remaining fields one recovers the more general gauge-fixing mechanism in Batalin-Vilkovisky theory which consists in arbitrary canonical transformations within the antibracket (see section 5.3 of [10]).

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[7]Note that it is *not* chosen to be a function of the difference $\phi^A - \varphi^A$, where φ^A is the collective field associated with ϕ^A.

Keeping the collective field also sheds some light on the fact that the Batalin-Vilkovisky scheme allows the quantization of theories with open algebras at all. How collective fields can be used to derive the Batalin-Vilkovisky scheme from [16] was shown in [15]. Precisely by introducing the collective field one can construct the BRST rules in such a way that the gauge fixing can be done as in the case of models with closed algebras, *i.e.* by adding $\delta\Psi(\phi,\phi^*)$ to the action.[8]

It is precisely this point of view which is useful when quantizing models with an open algebra in an $Sp(2)$-invariant way, since no recipe analogous to [16] is available for this case. Consider the following action:

$$S_{\text{ext}} = S^{BLT}(\phi - \varphi_+, \phi^*, \bar{\phi}) + \epsilon^{ab}\phi^*_{Ab}\pi^A_a + \bar{\phi}_A\lambda^A - \varphi^A_+ B_A + \frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi(\phi). \tag{38}$$

The $Sp(2)$ BRST rules that we have in mind here are slight generalisations of (8), taking into account the redefinitions of (11):

$$\begin{aligned}
\delta_a\phi^A &= \pi^A_a \\
\delta_a\varphi^A_+ &= \pi^A_a - \epsilon_{ab}\frac{\delta^l S^{BLT}(\phi - \varphi_+, \phi^*, \bar{\phi})}{\delta\phi^*_{Ab}} \\
\delta_a\pi^A_b &= \epsilon_{ab}\lambda^A \\
\delta_a\lambda^A &= 0 \\
\delta_a\bar{\phi}_A &= (-1)^{\epsilon_A+1}\phi^*_{Aa} \\
\delta_a\phi^*_{Ab} &= \epsilon_{ab}B_A \\
\delta_a B_A &= 0 \ .
\end{aligned} \tag{39}$$

Here we have introduced linear transformed B-fields $B_A = M_{AB}B^B$ to get rid of the M-matrix. It is straightforward to verify that S_{ext} of (38) is $Sp(2)$ BRST invariant ($\delta_a S_{\text{ext}} = 0$) under the rules (39), provided that S^{BLT} satisfies the classical part of the BLT master equation (20). When taking into account the Jacobian of the measure under (39), one obtains the full quantum master equation (20).

Since the $Sp(2)$ rules are always $Sp(2)$-nilpotent on functions of ϕ, this shows that independent of the structure of the algebra, be it an open or closed algebra, the gauge fixing can always be done in a manifestly $Sp(2)$ BRST invariant way by adding a term of the form $\frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi(\phi)$. It is precisely the presence of the collective field which allowed us to construct the $Sp(2)$ BRST rules (39). The question of whether a gauge-fixed action that leads to a gauge independent path integral can be constructed for open algebras, is then seen to be equivalent to the question of whether the master equation (20) can be solved for open algebras.

Finally one sees that when one keeps the collective field a manifest $Sp(2)$ BRST invariant gauge fixing can be done by adding $\frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi(\phi,\bar{\phi},\phi^*_a)$. This seems to be the $Sp(2)$ BRST equivalent of the gauge fixing by canonical transformation that is so powerful in the Batalin-Vilkovisky case.

3 Triplectic Quantization

We are now ready to show how triplectic quantization [7] with very specific boundary conditions can be derived from the general Master Equation (17). We shall start with the particular Ansatz (19), which we saw lead directly to the $Sp(2)$-symmetric formalism of Batalin, Lavrov and Tyutin. We shall later see, by "using the Schwinger-Dyson BRST symmetry twice", how this can be reformulated. In the course of these derivations we shall elucidate the rôle played by the gauge-fixing

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[8]See section 6.2 of ref. [10].

function X in triplectic quantization, and see the precise interplay between the "action part" W and this gauge-fixing function X.

To begin, let us consider the particular Ansatz (19) for the solution S. We have already discussed the extent to which we can show this is the most general consistent Ansatz. Certainly, one appealing feature of it is that the remaining solution for S^{BLT} is a function of $\phi, \bar\phi$ and ϕ^* only. In esssence, we have with this Ansatz split the full solution S into the part $S^{BLT}[\phi, \bar\phi, \phi^*]$, and a trivial part, whose only function is to set the fields $\bar\phi_A$ and the antifields ϕ^*_{Aa} equal to zero prior to any gauge fixings of the internal symmetries. The resulting Master Equation is then that of Batalin, Lavrov and Tyutin, and its biggest advantage is that it is independent of the variables π^A_a and λ^A. There are of course an unlimited number of different ways in which one can substitute the very same Ansatz (19), because one can lump part of the terms that only serve as to fix the fields $\bar\phi$ and ϕ^*_{Aa} to zero into the "action", at the cost of having another condition on the remainder. Each different substitution will correspond to seemingly different formulations. There will be new Master Equations for the action, new conditions on the remaining terms etc. Triplectic quantization is one of these. To see this, define

$$
W = S^{BLT}[\phi, \bar\phi, \phi^*] + \frac{1}{2}\epsilon^{ab}\phi^*_{Ab}\pi^A_a \tag{40}
$$

$$
X = \frac{1}{2}\epsilon^{ab}\phi^*_{Ab}\pi^A_a + \bar\phi_A\lambda^A + S_\chi . \tag{41}
$$

Here S_χ is the "real" gauge-fixing function of internal symmetries. If we do not wish to change the boundary conditions on the action S^{BLT}, this last term is required to be both $Sp(2)$-symmetric and exact, *i.e.*, of the form $S_\chi = \frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi$, but it is instructive to keep it like an apparently arbitrary function for the moment.

We first plug the definition (40) into the Master Equation (20). This gives

$$
-\left[\frac{\delta^r W}{\delta\phi^*_{Ac}} + \frac{1}{2}(-1)^{\epsilon_A}\epsilon^{ac}\pi^A_a\right]\frac{\delta^l W}{\delta\phi^A} - \epsilon^{ca}\frac{\delta^l W}{\delta\bar\phi_A}\phi^*_{Aa} = (i\hbar)\Delta^a W . \tag{42}
$$

This is a new and valid form of the Master Equation for W, but we next choose to rewrite it in the following way. From eq. (40) we see that

$$
\epsilon^{ab}\phi^*_{Ab} = 2\frac{\delta^r W}{\delta\pi^A_a} . \tag{43}
$$

We next – arbitrarily – use this identity only on one half of the ϕ^*_{Aa}-term on the l.h.s. of eq. (42), while keeping the remaining half. Introducing

$$
\hat{V}^c F \equiv \frac{1}{2}\epsilon^{ca}\left[(-1)^{\epsilon_A}\frac{\delta^r F}{\delta\bar\phi_A}\phi^*_{Aa} - \frac{\delta^r F}{\delta\phi^A}\pi^A_a\right] = -\frac{1}{2}V^c F - \frac{1}{2}\epsilon^{ca}\frac{\delta^r F}{\delta\phi^A}\pi^A_a , \tag{44}
$$

the resulting equation for W can be written

$$
\frac{1}{2}\{W, W\}^a - \hat{V}^a W = (i\hbar)\hat{\Delta}^a W . \tag{45}
$$

Here we have made use of the definition (3) and (4). This is the quantum Master Equation for the "action" W of triplectic quantization [7]. It is important to mention that with our explicit parametrization for W, eq. (40), the second term in the definition of $\hat\Delta^a$ from eq. (4) does not contribute, but of course can be added.

What about the remaining terms, which have been denoted by X? Using the definition (41) one immediately finds that for $S_\chi = 0$ this term satisfies

$$\frac{1}{2}\{X,X\}^a + \hat{V}^a X = (i\hbar)\hat{\Delta}^a X \ . \tag{46}$$

Here $\Delta^a X = 0$ with our explicit parametrization, so not only does the second term in eq. (4) not contribute (as is the case for W), this even holds for the first term as well. In other words, in the split $S = W + X$, all quantum corrections are absorbed in W. That this is possible is not too surprising, since indeed the whole split into W and X is arbitrary, and with the present choice we are simply automatically guaranteed that we only have to be concerned with quantum corrections to the "action" W. In fact, this is an appealing feature of this particular formulation, which, as we shall see shortly, can easily be lost.

So far we have only discussed the trivial terms in X, those in fact that have nothing to do with the gauge fixing of internal symmetries. We next reinstate the function S_χ as in eq. (41). If we still wish eq. (46) to be satisfied, we find that this function S_χ can be an *arbitrary* $Sp(2)$ BRST invariant function of ϕ, π and λ. Of course, if we wish that this term does not change the theory, *i.e.* the Schwinger-Dyson equations, this term *must* be $Sp(2)$ BRST exact. From our point of view: otherwise this term should be added to S on the r.h.s. of the Schwinger-Dyson BRST rules (13). From the point of view of triplectic quantization: this term will, if it is not $Sp(2)$ BRST exact, modify the boundary conditions on W and X. In this sense neither the Master Equation for W nor the corresponding one for X are meaningful before the boundary conditions are stipulated.

Gauge-fixing is thus, if we wish to remain within the present framework, restricted to gauge bosons χ which are functions of ϕ only. It is amusing that the "gauge fixing" function X also consists of terms which have nothing to do with the gauge fixing of internal symmetries. Similarly, the only way of "deriving" the gauge fixing function S_χ from solving the functional differential equation (46) is by imposing very specific boundary conditions. (These boundary conditions can easily be found in the present approach, because we simply have to insert the explicit form of a valid S_χ, and read off what valid boundary conditions will correspond to). This is because any $Sp(2)$ BRST exact term $\frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi(\phi)$ can be added to any solution X: the sum $X + \frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi(\phi)$ then satisfies the Master Equation for X trivially. We thus (naturally) derive no constraint on such a term by substituting the full solution into the Master Equation for X.

Thus, in this formulation X has the possibility of containing a true gauge-fixing function S_χ, but it will also contain additional terms. Of course, then, the main arbitrariness of X just corresponds to the freedom of adding $Sp(2)$ BRST exact terms. But such $Sp(2)$ BRST exact terms can also be added to W. It would clearly be advantageous if the two Master Equations for W and X could be chosen so that X *precisely* equals the gauge-fixing term S_χ, but this, as we will discuss below, is unfortunately impossible by construction.

We would like to emphasize that the particular split (40,41) of course is completely arbitrary. We could absorb an arbitrary fraction of the $\epsilon^{ab}\phi^*_{Ab}\pi^A_b$-term into W (leaving the rest in X), and also absorb an arbitrary fraction of the $\bar{\phi}_A\lambda^A$-term into W, again leaving the rest in X. Similarly, in the way we made use of the identity (43) for ϕ^*_{Aa}, we could of course have chosen not to employ this identity at all, or to employ it with a different fraction. All of these choices will be completely equivalent in the end, but at intermediate stages they correspond to different definitions of the antibracket and the vector field. In this manner we see very explicitly the arbitrariness of formulation which we mentioned in the Introduction, and which was also emphasized in refs. [2, 5].

The impossibility of constructing Master Equations for W and X in such a manner that X really equals the gauge-fixing function (and no other terms), and in such a manner that the equation would give non-trivial conditions on this gauge-fixing term, is now obvious. This would correspond to a

split

$$W \equiv S^{BLT}[\phi, \bar{\phi}, \phi^*] + \epsilon^{ab}\phi^*_{Ab}\pi^A_b + \bar{\phi}_A\lambda^A \tag{47}$$

$$X \equiv S_\chi \tag{48}$$

instead of eq. (40,41). But then W is simply the full action S. And the equation for X has entirely disappeared, because an $Sp(2)$ BRST exact term can always be added to S without any consequences. So by construction it is impossible to achieve a split of the full action S into W and X in such a way that X is the gauge-fixing function S_χ, and nothing else. This is also obvious from the start, since an $Sp(2)$ BRST exact term $S_\chi = \frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi(\phi)$ trivially satisfies $\delta_c S_\chi = 0$, for any $\chi(\phi)$.[9]

So far we have succeeded in directly deriving from first principles the method of triplectic quantization, and we have shown how the gauge-fixing mechanism appears in this formalism. We have managed to split the parametrization (40,41) up in such a manner that $S = W + X$, and both W and X satisfy the Master Equations of triplectic quantization. The full path integral is invariant under the associated $Sp(2)$ BRST transformations (13).

However, there are still a few points of difference with the formulation of ref. [7]. In particular, Batalin and Marnelius [7] do not impose so strong boundary conditions on W and X that the only solutions are those corresponding to the split (40,41). As one consequence, triplectic quantization can lead to solutions X that acquire quantum corrections, a complication which is avoided in the way we have formulated it above. Moreover, the $Sp(2)$ BRST symmetry of the full path integral in triplectic quantization appears to be different from ours, that of triplectic quantization being represented by [7]:

$$\hat{\delta}^a F = \{F, -W + X\}^a + 2\hat{V}^a F \tag{49}$$

This differs from our Schwinger-Dyson BRST symmetry in a manner to be specified below. In addition, as we already mentioned in the Introduction, the BRST symmetry (49) seems to depend on the gauge fixing even in the most simple and known cases of closed irreducible gauge algebras. There are thus still a few points of difference between the general $Sp(2)$-covariant Master Equation (17), the Ansatz (19) for the solution S, the split (40,41), and the complete formalism of triplectic quantization. We shall now explain the origin of these discrepancies.

As a first step, let us focus on the difference between our BRST transformations (13) and those of triplectic quantization (49). Let us first use the explicit form (40,41) of W and X, omitting the in this context unimportant gauge-fixing term S_χ (see below). We write out the transformations (49) in detail, using the definition (40,41):

$$\hat{\delta}^a\phi^A = -\epsilon^{ab}\pi^A_b - \frac{\delta^l S^{BLT}}{\delta\phi^*_{Aa}}$$

$$\hat{\delta}^a\bar{\phi}_A = (-1)^{\epsilon_A}\epsilon^{ab}\phi^*_{Ab}$$

$$\hat{\delta}^a\pi^A_b = -\delta^a_b\lambda^A + \delta^a_b\frac{\delta^l S^{BLT}}{\delta\bar{\phi}_A}$$

$$\hat{\delta}^a\phi^*_{Ab} = \delta^a_b\frac{\delta^l S^{BLT}}{\delta\phi^A}$$

$$\hat{\delta}^a\lambda^A = 0 . \tag{50}$$

Comparing with our eq. (13) which gives the Schwinger-Dyson BRST symmetry δ^a, we note that except for the way $\hat{\delta}^a$ acts on ϕ^A and π^A_b, we have $\hat{\delta}^a = -\delta^a$. Our task is then to first explain

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[9]And as we have discussed earlier, this even holds if χ is an arbitrary function of ϕ^A, $\bar{\phi}_A$, π^A_a and λ^A.

how these additional terms in the transformation laws for ϕ^A and π_b^A can be allowed. To this end, consider the following identities, valid before we add a gauge-fixing terms S_χ:

$$\epsilon^{ab}\phi_{Ab}^* \;=\; \frac{1}{2}\left[\epsilon^{ab}\phi_{Ab}^* + (-1)^{\epsilon_A+1}\frac{\delta^l S}{\delta\pi_a^A}\right] \tag{51}$$

$$\frac{\delta^r S}{\delta\phi_{Ab}^*} \;=\; \frac{\delta^r S^{BLT}}{\delta\phi_{Ab}^*} + (-1)^{\epsilon_A+1}\epsilon^{ab}\pi_b^A \tag{52}$$

$$\frac{\delta^r S}{\delta\bar\phi_A} \;=\; \frac{\delta^r S^{BLT}}{\delta\bar\phi_A} + (-1)^{\epsilon_A}\lambda^A \; . \tag{53}$$

Let us now see what classical BRST invariance of the action implies. Using the original transformations (13), we get

$$0 = \delta^a S = \frac{\delta^r S}{\delta\phi^A}\epsilon^{ab}\pi_b^A + \frac{\delta^r S}{\delta\bar\phi_A}(-1)^{\epsilon_A+1}\epsilon^{ab}\phi_{Ab}^* - \frac{\delta^r S}{\delta\phi_{Aa}^*}\frac{\delta^l S}{\delta\phi^A} + \frac{\delta^r S}{\delta\pi_a^A}\lambda^A \; . \tag{54}$$

We next substitute eqs. (51-53), and get, successively:

$$\begin{aligned}
0 \;=\; & \frac{\delta^r S}{\delta\phi^A}\epsilon^{ab}\pi_b^A + \frac{1}{2}\frac{\delta^r S}{\delta\bar\phi_A}(-1)^{\epsilon_A+1}\epsilon^{ab}\phi_{Ab}^* + \frac{1}{2}\frac{\delta^r S}{\delta\bar\phi_A}\frac{\delta^l S}{\delta\pi_a^A} - \frac{1}{2}\frac{\delta^r S}{\delta\phi_{Aa}^*}\frac{\delta^l S}{\delta\phi^A} \\
& -\frac{1}{2}\left[\frac{\delta^r S^{BLT}}{\delta\phi_{Aa}^*} + (-1)^{\epsilon_A+1}\epsilon^{ab}\pi_a^A\right]\frac{\delta^l S}{\delta\phi^A} + \frac{\delta^r S}{\delta\pi_a^A}\lambda^A \\
\;=\; & \frac{\delta^r S}{\delta\phi^A}\left[\frac{1}{2}\epsilon^{ab}\pi_b^A + \frac{1}{2}\frac{\delta^l S^{BLT}}{\delta\phi_{Aa}^*}\right] + \frac{\delta^r S}{\delta\bar\phi_A}\left[\frac{1}{2}(-1)^{\epsilon_A+1}\epsilon^{ab}\phi_{Ab}^*\right] \\
& +\frac{\delta^r S}{\delta\pi_a^A}\left[\frac{1}{2}\lambda^A - \frac{1}{2}\frac{\delta^l S^{BLT}}{\delta\bar\phi_A}\right] + \frac{\delta^r S}{\delta\phi_{Aa}^*}\left[-\frac{1}{2}\frac{\delta^l S}{\delta\phi^A}\right]
\end{aligned} \tag{55}$$

We see from these rewritings that at the classical level a completely equivalent BRST-like symmetry of the action is represented by

$$\begin{aligned}
\tilde\delta^a\phi^A \;&=\; \epsilon^{ab}\pi_b^A + \frac{\delta^l S^{BLT}}{\delta\phi_{Aa}^*} \\
\tilde\delta^a\bar\phi_A \;&=\; (-1)^{\epsilon_A+1}\epsilon^{ab}\phi_{Ab}^* \\
\tilde\delta^a\pi_b^A \;&=\; \delta_b^a\lambda^A - \delta_b^a\frac{\delta^l S^{BLT}}{\delta\bar\phi_A} \\
\tilde\delta^a\phi_{Ab}^* \;&=\; -\delta_b^a\frac{\delta^l S}{\delta\phi^A} \\
\tilde\delta^a\lambda^A \;&=\; 0 \; .
\end{aligned} \tag{56}$$

In the transformation law for ϕ_{Aa}^* we can substitute S by S^{BLT} because these two actions differ only by terms independent of ϕ^A. We then have $\tilde\delta^a = -\hat\delta^a$, and we have recovered the transformation rules (50) of Batalin and Marnelius [7]. It is straightforward to check that adding a gauge-fixing term S_χ does not alter this conclusion, although of course both (50) and (56) now contain some additional pieces.

The relation between the symmetries (49) and (13) may also be illuminated by considering the way in which a Ward Identity of the kind $\langle\delta^a G[\phi,\bar\phi,\pi,\phi^*,\lambda]\rangle = 0$ may be rewritten by means of partial integrations inside the path integral. First notice that the original Schwinger-Dyson $Sp(2)$ BRST algebra (13) can be rewritten by means of the explicit form (40,41) of W and X:

$$\delta^a\phi^A \;=\; \frac{1}{2}\epsilon^{ab}\pi_b^A - \frac{\delta^l X}{\delta\phi_{Aa}^*}$$

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$$\delta^a \bar{\phi}_A = \frac{1}{2}(-1)^{\epsilon_A+1}\epsilon^{ab}\phi^*_{Ab} + \frac{\delta^l W}{\delta\pi^A_a}$$

$$\delta^a \pi^A_b = \delta^a_b \frac{\delta^l X}{\delta\bar{\phi}_A}$$

$$\delta^a \phi^*_{Ab} = -\delta^a_b \frac{\delta^l W}{\delta\phi^A}$$

$$\delta^a \lambda^A = 0 . \tag{57}$$

For an arbitrary function $G = G[\phi, \bar{\phi}, \pi, \phi^*, \lambda]$ we then have:

$$
\begin{aligned}
\delta^a G &= \frac{\delta^r G}{\delta\phi^A}\left(\frac{1}{2}\epsilon^{ab}\pi^A_b - \frac{\delta^l X}{\delta\phi^*_{Aa}}\right) + \frac{\delta^r G}{\delta\bar{\phi}_A}\left(\frac{1}{2}(-1)^{\epsilon_A+1}\epsilon^{ab}\phi^*_{Ab} + \frac{\delta^l W}{\delta\pi^A_a}\right) \\
&\quad + \frac{\delta^r G}{\delta\pi^A_a}\frac{\delta^l X}{\delta\bar{\phi}_A} - \frac{\delta^r G}{\delta\phi^*_{Aa}}\frac{\delta^l W}{\delta\phi^A} \\
&= -\frac{\delta^r G}{\delta\phi^A}\frac{\delta^l X}{\delta\phi^*_{Aa}} - \frac{\delta^r G}{\delta\phi^*_{Aa}}\frac{\delta^l W}{\delta\phi^A} + \frac{\delta^r G}{\delta\bar{\phi}_A}\frac{\delta^l W}{\delta\pi^A_a} + \frac{\delta^r G}{\delta\pi^A_a}\frac{\delta^l X}{\delta\bar{\phi}_A} - \hat{V}^a G \tag{58}
\end{aligned}
$$

Now put the above expression (58) inside the path integral. Let us first take a closer look at the first term in (58). A partial integration of the ϕ^*_{Aa}-derivative produces a similar term where X is replaced by $-W$, as well as a quantum correction. This is quite analogous to the manipulations that lead us to the quantum BRST operator in section 2.1. As we shall see below, in this case we shall be able to remove the quantum correction. In detail:

$$\int \mathcal{D}\mu \frac{\delta^r G}{\delta\phi^A}\frac{\delta^l X}{\delta\phi^*_{Aa}} e^{\frac{i}{\hbar}(W+X)} = \int \mathcal{D}\mu \left(\frac{\delta^r G}{\delta\phi^A}\frac{\delta^l(-W)}{\delta\phi^*_{Aa}} + (-1)^{\epsilon_A+1}i\hbar\frac{\delta^r}{\delta\phi^*_{Aa}}\frac{\delta^r G}{\delta\phi^A}\right) e^{\frac{i}{\hbar}(W+X)} , \tag{59}$$

where $\mathcal{D}\mu$ denotes the (flat) path integral measure. Actually, it turns out to be convenient to arbitrarily perform a partial integration on only *half* of this term, leaving the other half untouched:

$$\frac{\delta^r G}{\delta\phi^A}\frac{\delta^l X}{\delta\phi^*_{Aa}} \to \frac{1}{2}\frac{\delta^r G}{\delta\phi^A}\frac{\delta^l(X-W)}{\delta\phi^*_{Aa}} + \frac{1}{2}(-1)^{\epsilon_A+1}i\hbar\frac{\delta^r}{\delta\phi^*_{Aa}}\frac{\delta^r}{\delta\phi^A}G \tag{60}$$

where the arrow indicates that partial integrations are required inside the path integral. Doing analogous partial integrations on each of the first four terms in (58) gives

$$\delta^a G \to -\frac{1}{2}\{G, X-W\}^a - \hat{V}^a G = -\frac{1}{2}\hat{\delta}^a G , \tag{61}$$

where again the arrow indicates that partial integrations have been used. Remarkably, the four otherwise inescapable quantum terms have completely cancelled out. Apart from the irrelevant overall factor of 1/2, we have recovered the BRST-like transformations of triplectic quantization [7].

At this point one could worry that the symmetry (56) would lead to a different Master Equation when acting on an arbitrary action S of the form (19). This is, however, not the case. In detail, one finds that

$$\tilde{\delta}^a S = 2\frac{\delta^r S^{BLT}}{\delta\phi^A}\frac{\delta^l S^{BLT}}{\delta\phi^*_{Aa}} - 2\epsilon^{ab}\phi^*_{Ab}\frac{\delta^l S^{BLT}}{\delta\bar{\phi}_A} = 2i\hbar\Delta^a S^{BLT} , \tag{62}$$

i.e., precisely *twice* the Master Equation (20) for S^{BLT}. (We have here used that by construction S^{BLT} is independent of π^A_a.) In this precise sense the symmetry (56) corresponds to applying the $Sp(2)$ Schwinger-Dyson BRST transformations twice.

There is, however, one important difference between a formulation based on (56) and one based on (13). While the original BRST transformations (13) by construction obey the $Sp(2)$ algebra $\delta_a\delta_b + \delta_b\delta_a = 0$ when acting on all fields except the antifields ϕ^*_{Aa}, this is not the case for the version (50). In fact, the transformations (50) do not even obey this algebra when restricted to the subspace containing the original fields ϕ^A. This means that we are unable to gauge-fix in the usual simple manner by adding $Sp(2)$ BRST exact terms of the kind $\frac{1}{2}\epsilon^{ab}\delta_a\delta_b\chi(\phi)$.

We have thus seen that although it is inconvenient from the point of view of gauge fixing, we can equivalently work with the BRST-like symmetry (50) – provided, of course, that we restrict ourselves to the parametrization (40,41) for W and X. One way to go beyond this particular parametrization corresponds to adding $Sp(2)$ BRST exact terms to S^{BLT} beyond those required for the gauge fixing of internal gauge symmetries.

Triplectic quantization consists in its simplest form in elevating the status of eq. (49) to that of defining the BRST-like symmetry beyond any particular parametrization of W and X.[10] In the Master Equations for W and X the only new requirement is that it is then precisely the particular Δ-operator (4) which must be employed. Both W and X will then acquire quantum corrections in general.

It is now clear why the $Sp(2)$-covariant Schwinger-Dyson BRST symmetry is unable to provide a derivation of triplectic quantization in its most general form. Because as our starting point we have insisted that the $Sp(2)$ version of this Schwinger-Dyson BRST symmetry should obey the $Sp(2)$ generalization of nilpotency, $\delta_a\delta_b + \delta_b\delta_a = 0$. This $Sp(2)$ algebra of the BRST symmetry is not needed in order to ensure correct Schwinger-Dyson equations as BRST Ward Identities. While we could thus easily dispose of it, we have insisted on preserving it in order to have available a straightforward gauge-fixing mechanism, and in order to keep contact with BRST cohomology theory. The analogous situation in conventional Lagrangian BRST quantization would consist in abandoning nilpotency of the BRST operator, even on-shell. This is possible, because all the Lagrangian path integral is required to provide, are correct and well-defined Schwinger-Dyson equations. But we have here chosen on purpose to remain tied within the more stringent framework of nilpotent BRST operators. We have not seen any need to leave this framework either, and in particular we have been unable to prove that there are cases of gauge algebras which can be solved by triplectic quantization, and which cannot be solved by the method proposed earlier by Batalin, Lavrov and Tyutin [6].

4 Conclusions and Outlook

Our purpose has been to investigate to what extent the principle of imposing Schwinger-Dyson BRST symmetry, here in an $Sp(2)$-covariant form, can be used as an underlying principle for Lagrangian BRST quantization.

We have succeeded in deriving the direct analogue of the Master Equation (1) in $Sp(2)$-symmetric form. Just as the Ansatz $S[\phi, \phi^*, c] = S^{BV}[\phi, \phi^*] - \phi^*_A c^A$ reduces (1) to the Batalin-Vilkovisky Master Equation for S^{BV} [3], the Ansatz (19) reduces the $Sp(2)$ Master Equation (17) to the Master Equation for S^{BLT} proposed earlier by Batalin, Lavrov and Tyutin [6]. Throughout the $Sp(2)$-extended BRST algebra is fulfilled on all fields except the antifields ϕ^*_{Aa}, in complete analogy with the usual Lagrangian BRST quantization. Technically, this particular formulation is superior to that of ref. [5] because the $Sp(2)$ analogue of nilpotency can be kept even after removing the collective field φ^A_+. This makes gauge fixing completely straightforward in this formulation, and

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[10] In triplectic quantization only X is chosen to depend on the fields λ, but since these fields do not transform in the most simple formulation, also this condition can be relaxed.

analogous to gauge fixing in Batalin-Vilkovisky quantization.

Rewriting the same Ansatz that leads to the Batalin, Lavrov and Tyutin formalism in terms of W and X as in eq. (40,41), we have shown how this scheme fits into the language of the recently proposed $Sp(2)$-symmetric triplectic quantization of Batalin and Marnelius [7]. We have traced the origin of the more general equations of triplectic quantization to the lack of an $Sp(2)$-invariant BRST algebra of the form $\delta_a\delta_b + \delta_b\delta_a = 0$ in triplectic quantization. We have seen no need to abandon this algebra, and in particular we are not aware of any gauge algebras that cannot be treated within the more economic framework of Batalin, Lavrov and Tyutin. The Master Equation (20) ensures that all Schwinger-Dyson equations are correctly reproduced by the BRST Ward Identities, and preserves the $Sp(2)$ BRST algebra on all fields except ϕ^*_{Aa} (which is all that is needed in order to gauge-fix in the conventional Lagrangian manner).

There are, in our opinion, still a number of unanswered questions and new directions for research. The relation between the reformulation of Batalin-Vilkovisky quantization in the same language as triplectic quantization [8] deserves further study. Within the $Sp(2)$-symmetric framework it should be proved whether or not triplectic quantization can solve cases not covered by the method of Batalin, Lavrov and Tyutin.

Of current interest is also covariant formulations of these schemes. The covariant generalization of triplectic quantization has already been worked out in detail [8]. The covariant generalization of eq. (17), and in particular then also of eq. (20), can be derived using the technique described in refs. [13, 18]. Work in this direction is presently underway.

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